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Transmission Line Input Impedance Recursion Calculation.

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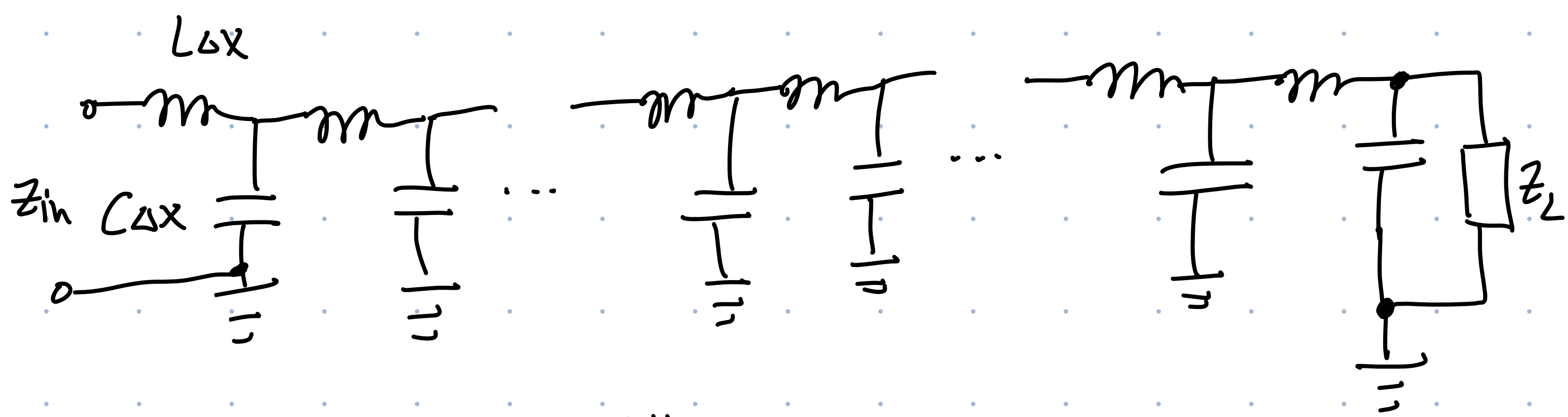
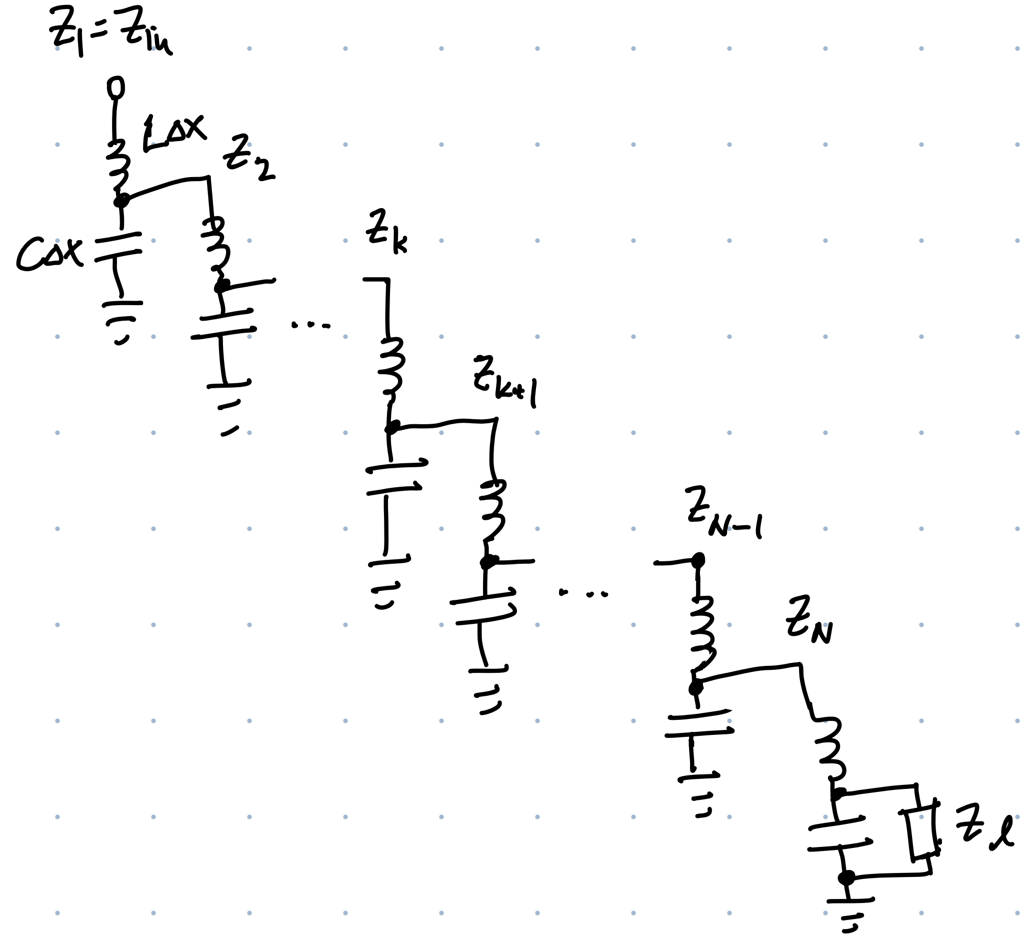
TL Circuit model.



Can redraw in the following way:



$$Z_N = Z_L + Z_C \| Z_\ell$$

define the load impedance as

$$Z_{N+1} = Z_\ell$$

$$Z_{N-1} = Z_L + Z_C \| Z_N$$

$$\vdots$$

$$\boxed{Z_k = Z_L + Z_C \| Z_{k+1}}$$

Impedance recursion relation.

$$Z_k = j\omega L \Delta x + \left(\frac{\frac{Z_{k+1}}{j\omega C \Delta x}}{\frac{1}{j\omega C \Delta x} + Z_{k+1}} \right)$$

$$\therefore Z_k = j\omega L \Delta x + \frac{Z_{k+1}}{1 + j\omega C \Delta x Z_{k+1}}$$

Note that we can write $Z_k = \dfrac{V_k}{I_k}$

$$Z_{k+1} = \frac{V_{k+1}}{I_{k+1}}$$

$$\therefore \frac{V_k}{I_k} = j\omega L \Delta x + \frac{V_{k+1}/I_{k+1}}{1 + j\omega C \Delta x \frac{V_{k+1}}{I_{k+1}}}$$

$$= \frac{j\omega L\Delta x\left(1 + j\omega C\Delta x \frac{V_{k+1}}{I_{k+1}}\right) + \frac{V_{k+1}}{I_{k+1}}}{1 + j\omega C\Delta x \frac{V_{k+1}}{I_{k+1}}}$$

$$= \frac{j\omega L\Delta x\left(I_{k+1} + j\omega C\Delta x V_{k+1}\right) + V_{k+1}}{I_{k+1} + j\omega C\Delta x V_{k+1}}$$

$$\therefore \frac{V_k}{I_k} = \frac{(1-\omega^2 LC\Delta x^2)V_{k+1} + j\omega L\Delta x I_{k+1}}{j\omega C\Delta x V_{k+1} + I_{k+1}}$$

We can express this relationship via the following "transfer" matrix:

$$\begin{pmatrix} V_k \\ I_k \end{pmatrix} = \underbrace{\begin{pmatrix} 1-\omega^2 LC\Delta x^2 & j\omega L\Delta x \\ j\omega C\Delta x & 1 \end{pmatrix}}_{\equiv \underline{\underline{T}}} \begin{pmatrix} V_{k+1} \\ I_{k+1} \end{pmatrix}$$

$$\begin{pmatrix} V_k \\ I_k \end{pmatrix} = \underline{\underline{T}} \begin{pmatrix} V_{k+1} \\ I_{k+1} \end{pmatrix}$$

The transfer matrix, or its inverse gives us a way to calc the voltage & current amplitudes at each node of the descrete TL.

The strategy is to start at the load where

$$Z_{N+1} = Z_\ell = \frac{V_\ell}{I_\ell}$$

Then, apply T repeatedly in order to work our way back to $Z_{in} = Z_1 = \frac{V_1}{I_1}$.

Need to apply $\underline{\underline{T}}$ N times to go from Z_{N+1} to Z_1.

$$\therefore \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \underline{\underline{T}}^N \begin{pmatrix} V_\ell \\ I_\ell \end{pmatrix}$$

Let's work on $\underline{\underline{T}}^N$

$$\underline{\underline{T}} = \begin{pmatrix} 1-\omega^2 LC\Delta x^2 & j\omega L\Delta x \\ j\omega C\Delta x & 1 \end{pmatrix}$$

First, in the limit of large N, $\Delta x = \dfrac{l}{N} \to 0$.

$\therefore$ we'll neglect Δx^2.

$$\underline{\underline{T}} \simeq \begin{pmatrix} 1 & j\omega L\Delta x \\ j\omega C\Delta x & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\underline{\underline{I}}} + j\omega\Delta x \underbrace{\begin{pmatrix} 0 & L \\ C & 0 \end{pmatrix}}_{\equiv \underline{\underline{A}}}$$

$$\boxed{\underline{\underline{T}} = \underline{\underline{I}} + j\omega\Delta x\, \underline{\underline{A}}}$$

Now, $\underline{\underline{T}}^N = \left(\underline{\underline{I}} + j\omega\Delta x\, \underline{\underline{A}}\right)^N$

but, recall, $\Delta x = \dfrac{l}{N}$ where l is the TL length.

$$\underline{\underline{T}}^N = \left(\underline{\underline{I}} + \frac{j\omega l \underline{\underline{A}}}{N} \right)^N$$

Recall the scalar identity

$$\lim_{N\to\infty} \left(1 + \frac{x}{N} \right)^N = e^x$$

Can get to this relation via a compound interest calculation $\rightsquigarrow$ see Numberphile.

We will apply this identity to our matrix equation w/ $x \to j\omega l \underline{\underline{A}}$ and $\underline{\underline{A}} = \begin{pmatrix} 0 & L \\ C & 0 \end{pmatrix}$

$$\therefore \underline{\underline{T}}^N = e^{j\omega l \underline{\underline{A}}}$$

To evaluate the exponential of a matrix, we use the series representation of $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$

Before doing so, however, note the following:

$$\underline{\underline{A}}^2 = \begin{pmatrix} 0 & L \\ C & 0 \end{pmatrix}\begin{pmatrix} 0 & L \\ C & 0 \end{pmatrix} = \begin{pmatrix} LC & 0 \\ 0 & LC \end{pmatrix} = LC\,\underline{\underline{I}}$$

identity matrix

$$\underline{T}^N = \underline{\underline{I}} + j\omega l\,\underline{\underline{A}} + \frac{1}{2!}\underbrace{(j\omega l)^2}_{-(\omega l)^2}\underbrace{\underline{\underline{A}}^2}_{LC\underline{\underline{I}}} + \frac{1}{3!}\underbrace{(j\omega l)^3}_{-j(\omega l)^3}\underbrace{\underline{\underline{A}}^3}_{LC\underline{\underline{A}}} + \frac{1}{4!}\underbrace{(j\omega l)^4}_{(\omega l)^4}\underbrace{\underline{\underline{A}}^4}_{(LC)^2\underline{\underline{I}}} + \dots$$

$$= \underline{\underline{I}}\left[1 - \frac{1}{2!}\left(\omega\sqrt{LC}\,l\right)^2 + \frac{1}{4!}\left(\omega\sqrt{LC}\,l\right)^4 - \dots\right]$$

cos βl

$$j\left(\frac{1}{\sqrt{LC}}\right)\underline{\underline{A}}\left[\omega\sqrt{LC}\,l - \frac{1}{3!}\left(\omega\sqrt{LC}\,l\right)^3 + \dots\right]$$

sin βl

If we identify $\omega\sqrt{LC} \equiv \beta$, we can write:

$$\underline{\underline{T}}^N = \underline{\underline{I}}\cos\beta l + j\,\underline{\underline{A}}'\sin\beta l$$

Note that $A' = \frac{1}{\sqrt{LC}} \begin{pmatrix} 0 & L \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\frac{L}{C}} \\ \sqrt{\frac{C}{L}} & 0 \end{pmatrix}$

Now, identify $Z_0 = \sqrt{\frac{L}{C}}$, such that:

$$A' = \begin{pmatrix} 0 & Z_0 \\ Z_0^{-1} & 0 \end{pmatrix}$$

Note that $\beta = \omega\sqrt{LC}$ is the TL prop. constant.

$Z_0 = \sqrt{\frac{L}{C}}$ is the TL's characteristic impedance.

Recall that L and C are the per-unit-length inductance and capacitance respectfully.

Finally, we can attempt to calc. Z_{in}.

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \underline{\underline{T}}^N \begin{pmatrix} V_\ell \\ I_\ell \end{pmatrix}$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\beta\ell + j \begin{pmatrix} 0 & Z_0 \\ Z_0^{-1} & 0 \end{pmatrix} \sin\beta\ell \right] \begin{pmatrix} V_\ell \\ I_\ell \end{pmatrix}$$

$$= \begin{pmatrix} V_\ell & 0 \\ 0 & I_\ell \end{pmatrix} \cos\beta\ell + j \begin{pmatrix} Z_0 I_\ell \\ V_\ell / Z_0 \end{pmatrix} \sin\beta\ell$$

$$\therefore \; V_1 = V_\ell \cos\beta\ell + j Z_0 I_\ell \sin\beta\ell$$

$$I_1 = I_\ell \cos\beta\ell + j \frac{V_\ell}{Z_0} \sin\beta\ell$$

$$Z_{in} = \frac{V_1}{I_1} = \frac{V_\ell \cos\beta\ell + j Z_0 I_\ell \sin\beta\ell}{I_\ell \cos\beta\ell + j \frac{V_\ell}{Z_0} \sin\beta\ell}$$

mult. top & btm by $\frac{1}{I_\ell \cos\beta\ell}$

$$Z_{in} = \frac{V_\ell / I_\ell + j Z_0 \tan \beta \ell}{1 + j \frac{V_\ell / I_\ell}{Z_0} \tan \beta \ell}$$

$$\begin{pmatrix} 1 & j\omega L \Delta x \\ j\omega C \Delta x & 1 \end{pmatrix} \begin{pmatrix} 1 & j\omega L \Delta x \\ j\omega C \Delta x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2j\omega L \Delta x \\ 2j\omega C \Delta x & 1 \end{pmatrix}$$

use that $V_\ell / I_\ell \equiv Z_\ell$ and mult. top & btm by Z_0

$$\boxed{Z_{in} = Z_0 \frac{Z_\ell + j Z_0 \tan \beta \ell}{Z_0 + j Z_\ell \tan \beta \ell}}$$

Expected result for the input impedance of a lossless TL of length ℓ.

Let's see if we can determine $V(x)$ & $I(x)$ at any point along a continuous TL.

Start w/

$$\begin{pmatrix} V_k \\ I_k \end{pmatrix} = \underline{\underline{T}} \begin{pmatrix} V_{k+1} \\ I_{k+1} \end{pmatrix}$$

$\underline{\underline{T}}$ propagates V & I backwards along the line.

$$\therefore \begin{pmatrix} V_{k+1} \\ I_{k+1} \end{pmatrix} = \underline{\underline{T}}^{-1} \begin{pmatrix} V_k \\ I_k \end{pmatrix}$$ ... forward propagating

$$\underline{\underline{T}}^{-1} = \left(\underline{\underline{I}} + j\omega \Delta x \, \underline{\underline{A}} \right)^{-1}$$

$$= \begin{pmatrix} 1 & j\omega\Delta x L \\ j\omega\Delta x C & 1 \end{pmatrix}^{-1}$$

For a 2x2 matrix $\underline{\underline{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\underline{\underline{B}}^{-1} = \frac{1}{\det B} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\therefore \underline{\underline{T}}^{-1} = \frac{1}{1 + \omega^2 \Delta x^2 LC} \begin{pmatrix} 1 & -j\omega\Delta x L \\ -j\omega\Delta x C & 1 \end{pmatrix}$$

ignore Δx^2 term.

$\therefore \underline{\underline{T}}^{-1} \approx \underline{\underline{I}} - j\omega\Delta x \, \underline{\underline{A}}$

$\therefore$ If $x = k\Delta x$ (step k times to a position $x = k\Delta x$) $\Rightarrow \Delta x = \frac{x}{k}$

then

$$\begin{pmatrix} V_k \\ I_k \end{pmatrix} = \left(\underline{\underline{T}}^{-1}\right)^k \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

$$= \left(\underline{\underline{I}} - \frac{j\omega x \underline{\underline{A}}}{k}\right)^k \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

If we make $\Delta x \to 0$, then need $k \to \infty$ steps to get to position x.

$$\therefore \begin{pmatrix} V(x) \\ I(x) \end{pmatrix} = e^{-j\omega x \underline{\underline{A}}} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

Same strategy as before... Write out series expansion of e^x.

Recall $\underline{\underline{A}}^2 = LC\,\underline{\underline{I}}$

and $\beta = \omega\sqrt{LC}$ $\quad Z_0 = \sqrt{\frac{L}{C}}$

$$e^{-j\omega x \underline{\underline{A}}} = 1 + \underbrace{\left(-j\omega x \underline{\underline{A}}\right)}_{-j\omega x \underline{\underline{A}}} + \frac{1}{2!}\underbrace{\left(-j\omega x \underline{\underline{A}}\right)^2}_{-(\omega x)^2 LC\,\underline{\underline{I}}}$$

$$+ \frac{1}{3!}\underbrace{\left(-j\omega x \underline{\underline{A}}\right)^3}_{j(\omega x)^3 LC\,\underline{\underline{A}}} + \frac{1}{4!}\underbrace{\left(-j\omega x \underline{\underline{A}}\right)^4}_{j(\omega x)^4 (LC)^2\,\underline{\underline{I}}} + \frac{1}{5!}\underbrace{\left(-j\omega x \underline{\underline{A}}\right)^5}_{-j(\omega x)^5 (LC)^2\,\underline{\underline{A}}} + \ldots$$

$$\therefore e^{-j\omega x \underline{\underline{A}}} = \underline{\underline{I}} \underbrace{\left[1 - \frac{(\beta x)^2}{2!} + \frac{(\beta x)^4}{4!} - \cdots \right]}_{\cos \beta x}$$

$$\underbrace{\frac{-j\underline{\underline{A}}}{\sqrt{LC}}}_{-j\underline{\underline{A}}'} \underbrace{\left[\beta x - \frac{(\beta x)^3}{3!} + \frac{(\beta x)^5}{5!} - \cdots \right]}_{\sin \beta x}$$

where $\underline{\underline{A}}' = \begin{pmatrix} 0 & Z_0 \\ Z_0^{-1} & 0 \end{pmatrix}$

$$\therefore e^{-j\omega x \underline{\underline{A}}} = \underline{\underline{I}} \cos \beta x - j \underline{\underline{A}}' \sin \beta x$$

$$\therefore \begin{pmatrix} V(x) \\ I(x) \end{pmatrix} = \left[\cos \beta x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - j \sin \beta x \begin{pmatrix} 0 & Z_0 \\ Z_0^{-1} & 0 \end{pmatrix} \right] \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

$$\therefore \; V(x) = V_1 \cos\beta x - j Z_0 I_1 \sin\beta x \quad (a)$$

$$I(x) = -j \frac{V_1}{Z_0} \sin\beta x + I_1 \cos\beta x \quad (b)$$

This looks different than $V(x) = V_+ e^{-j\beta x} + V_- e^{j\beta x}$ (c)

$$I(x) = \frac{1}{Z_0}\left[V_+ e^{-j\beta x} - V_- e^{j\beta x}\right] \quad (d)$$

Let's investigate.

TL geometry:

Start by find V_1 & I_1 from (c) & (d).

$$V_1 = V(0) = V_+ + V_-$$

$$I_1 = I(0) = \frac{1}{Z_0}\left[V_+ - V_-\right]$$

$$2V_+ = V_1 + Z_0 I_1$$

$$2V_- = V_1 - Z_0 I_1$$

$$\therefore V_+ = \frac{1}{2}\left(V_1 + Z_0 I_1\right)$$

$$V_- = \frac{1}{2}\left(V_1 - Z_0 I_1\right)$$

sub into c & d:

$$V(x) = \frac{1}{2}\left(V_1 + Z_0 I_1\right)e^{-j\beta x} + \frac{1}{2}\left(V_1 - Z_0 I_1\right)e^{j\beta x}$$

$$= \frac{V_1}{2}\left(e^{-j\beta x} + e^{j\beta x}\right) + j\frac{Z_0 I_1}{2j}\left(e^{-j\beta x} - e^{j\beta x}\right)$$

$$= V_1 \cos\beta x - j Z_0 I_1 \sin\beta x$$ same as a ✓

$$I(x) = \frac{1}{Z_0}\left[\frac{1}{2}(V_1 + Z_0 I_1)e^{-j\beta x} - \frac{1}{2}(V_1 - Z_0 I_1)e^{j\beta x}\right]$$

$$= \frac{1}{Z_0}\left[j\frac{V_1}{2j}\left(e^{-j\beta x} - e^{j\beta x}\right) + \frac{Z_0 I_0}{2}\left(e^{-j\beta x} + e^{j\beta x}\right)\right]$$

$$\therefore\ I(x) = -j\frac{V_1}{Z_0}\sin\beta x + I_0\cos\beta x \quad \text{same as } (b) \checkmark$$

The transfer matrix can be used to reproduce the wave equation that we derived in class.

Start from:

$$\begin{pmatrix} V_{k+1} \\ I_{k+1} \end{pmatrix} = \underline{\underline{T}}^{-1}\begin{pmatrix} V_k \\ I_k \end{pmatrix}$$

and:

$$\underline{\underline{T}}^{-1} \approx \underline{\underline{I}} - j\omega\Delta x\,\underline{\underline{A}}$$

$$\therefore \begin{pmatrix} V(x+\Delta x) \\ I(x+\Delta x) \end{pmatrix} = \left[\underline{\underline{I}} - j\omega \Delta x \underline{\underline{A}} \right] \begin{pmatrix} V(x) \\ I(x) \end{pmatrix}$$

$$\therefore \begin{pmatrix} \dfrac{V(x+\Delta x) - V(x)}{\Delta x} \\ \\ \dfrac{I(x+\Delta x) - I(x)}{\Delta x} \end{pmatrix} = -j\omega \underline{\underline{A}} \begin{pmatrix} V(x) \\ I(x) \end{pmatrix}$$

In the limit that $\Delta x \to 0$

$$\boxed{\begin{aligned} \frac{dV}{dx} &= -j\omega C I \\ \frac{dI}{dx} &= -j\omega L V \end{aligned}}$$

these lead to the familiar wave equations that we saw in class.